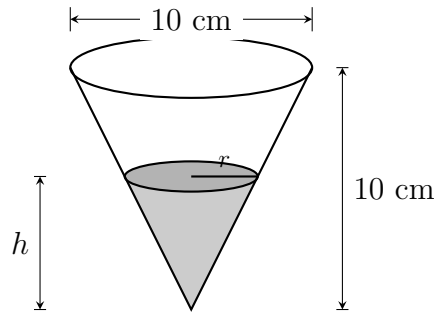


- A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $-\frac{3}{10}$ cm/hr.



(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

- Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.
- Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.
- Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

Solutions:

(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi \left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi \text{ cm}^3$

(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$
 $V = \frac{1}{3}\pi \left(\frac{1}{4}h^2\right) h = \frac{1}{12}\pi h^3$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$ therefore $\frac{dV}{dt}\Big|_{h=5} = \frac{1}{4}\pi(25) \left(-\frac{3}{10}\right) = -\frac{15}{8}\pi \text{ cm}^3/\text{hr}$

OR

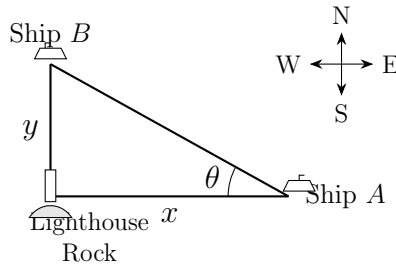
$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{3}\pi \left(r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right); \frac{dr}{dt} = \frac{1}{2} \frac{dh}{dt} \\ \frac{dV}{dt}\Big|_{h=5, r=\frac{5}{2}} &= \frac{1}{3}\pi \left(\left(\frac{25}{4}\right) \left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right) 5 \left(-\frac{3}{20}\right) \right) \\ &= -\frac{15}{8}\pi \text{ cm}^3/\text{hr} \end{aligned}$$

(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} = -\frac{3}{40}\pi h^2$
 $= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10} \cdot \text{area}$

The constant of proportionality is $-\frac{3}{10}$.

units of cm^3 in (a) and cm^3/hr in (b)

- Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t , and let y be the distance between Ship B and Lighthouse Rock at time t , as shown in the figure above.



- Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.
- Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.
- Let θ be the angle shown in the figure. Find the rate of change of θ , in radians per hour, when $x = 4$ km and $y = 3$ km.

Solutions:

- Distance = $\sqrt{3^2 + 4^2} = 5$ km
- $$r^2 = x^2 + y^2$$

$$2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \text{ or explicitly:}$$

$$r = \sqrt{x^2 + y^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2 + y^2}} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right)$$

At $x = 4, y = 3,$

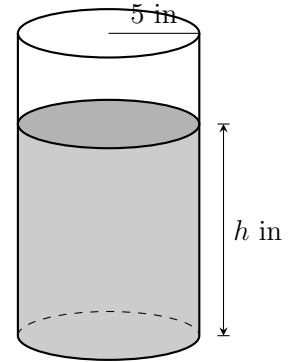
$$\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6 \text{ km/hr}$$
- $$\tan \theta = \frac{y}{x} \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

At $x = 4$ and $y = 3, \sec \theta = \frac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25} \left(\frac{10(4) - (-15)(3)}{16} \right)$$

$$= \frac{85}{25} = \frac{17}{5} \text{ radians/hr}$$

- A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t , measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



- (a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.
- (b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t .
- (c) At what time t is the coffeepot empty?

Solutions:

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}} dh = -\frac{1}{5} dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

- Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)
 - At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?
 - A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.
 - By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

Solution:

(a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000$ cm³/min

and $\frac{dr}{dt} = 2.5$ cm/min.

$$\begin{aligned}\frac{dV}{dt} &= 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt} \\ 2000 &= 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt} \\ \frac{dh}{dt} &= 0.038 \text{ or } 0.039 \text{ cm/min}\end{aligned}$$

(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$.

This occurs when $t = 25$ minutes.

Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$,

the oil slick reaches its maximum volume 25 minutes after the device begins working.

(c) The volume of oil, in cm³, in the slick at time $t = 25$ minutes

is given by $60,000 + \int_0^{25} (2000 - R(t)) dt$.

Problems adapted from the College Board Question Bank and released practice tests.