t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of

$$\int_{60}^{135} f(t)\,dt.$$

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by

$$g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$$

for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

---

a) $\int_{60}^{135} f(t)\,dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.

$$\begin{aligned}\int_{60}^{135} f(t)\,dt &\approx f(90)(90-60) + f(120)(120-90) + f(135)(135-120) \\ &= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25\end{aligned}$$

$\int_{60}^{135} f(t)\,dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.

$$\begin{aligned}\int_{60}^{135} f(t)\,dt &\approx f(90)(90-60) + f(120)(120-90) + f(135)(135-120) \\ &= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25\end{aligned}$$

b) f is differentiable. $\Rightarrow f$ is continuous on $[60, 120]$.

$$\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$$

By the Mean Value Theorem, there must exist a c, for $60 < c < 120$, such that $f'(c) = 0$.

c) $\frac{1}{150 - 0} \int_0^{150} g(t)dt$
The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).

d) $g'(140) \approx -0.004908$

$g'(140) = -0.005$ (or -0.004)

The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.

2. Stephen swims back and forth along a straight path in a 50-meter-long pool for 90 seconds. Stephen's velocity is modeled by $v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$, where t is measured in seconds and $v(t)$ is measured in meters per second.

(a) Find all times t in the interval $0 < t < 90$ at which Stephen changes direction. Give a reason for your answer.

(b) Find Stephen's acceleration at time $t = 60$ seconds. Show the setup for your calculations, and indicate units of measure. Is Stephen speeding up or slowing down at time $t = 60$ seconds? Give a reason for your answer.

(c) Find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. Show the setup for your calculations.

(d) Find the total distance Stephen swims over the time interval $0 \le t \le 90$ seconds. Show the setup for your calculations.

---

a) For $0 < t < 90$, $v(t) = 0 \Rightarrow t = 56$.

Stephen changes direction when his velocity changes sign. This occurs at $t = 56$ seconds.

b) $v'(60) = a(60) = -0.0360162$ Stephen's acceleration at time $t = 60$ seconds is -0.036 meter per second per second.

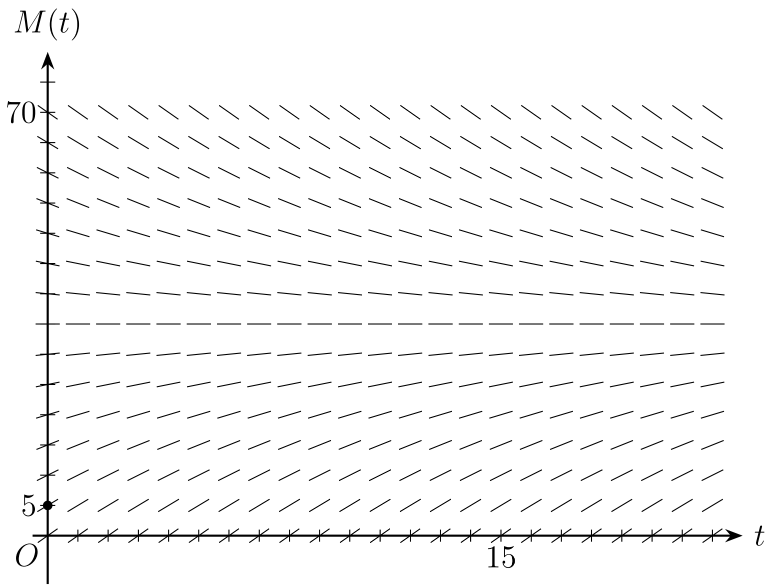
$v(60) = -0.1595124 < 0$ Stephen is speeding up at time $t = 60$ seconds because Stephen's velocity and acceleration are both negative at that time.

c) $\int_{20}^{80} v(t)dt = 23.383997$
The distance between Stephen's positions at $t = 20$ seconds and $t = 80$ seconds is 23.384 (or 23.383) meters.

d) $\int_{0}^{90} |v(t)|dt = 62.1642$
The total distance Stephen swims over the time interval $0 \le t \le 90$ seconds is 62.164 meters.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.

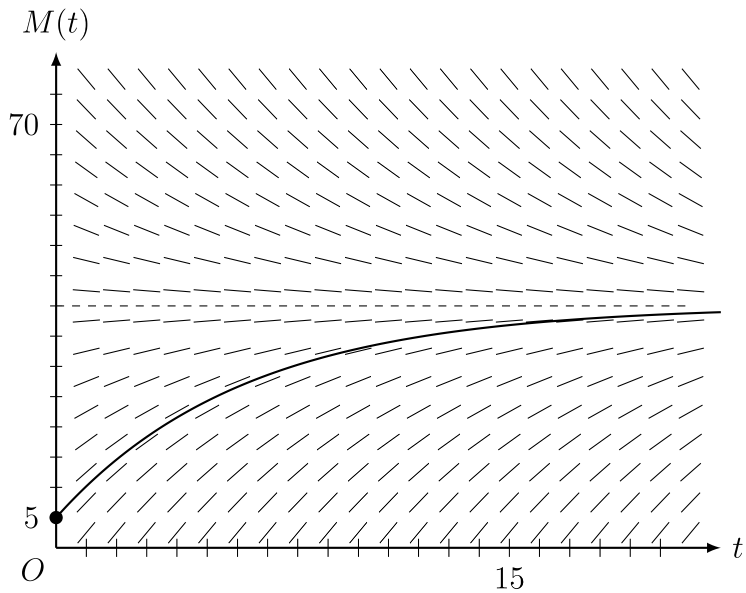


(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

a)



b)

$$\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$$

The tangent line equation is $y = 5 + \frac{35}{4}(t-0)$.

$M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$

The temperature of the milk at time $t = 2$ minutes is approximately 22.5° Celsius.

c)

$$\frac{d^2M}{dt^2} = \frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40-M)\right) = -\frac{1}{16}(40-M)$$

Because $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.

d)

$$\frac{dM}{40-M} = \frac{1}{4}dt$$

$$\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt$$

$$-\ln|40-M| = \frac{1}{4}t + C$$

$-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$

$M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow |40 - M| = 40 - M$

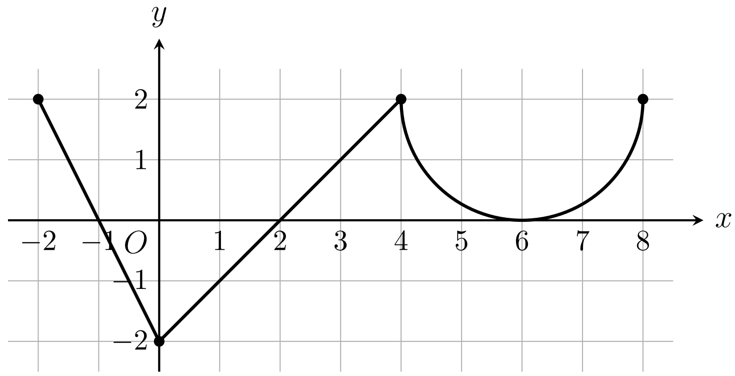
$-\ln(40 - M) = \dfrac{1}{4}t - \ln 35$

$\ln(40 - M) = -\dfrac{1}{4}t + \ln 35$

$40 - M = 35e^{-t/4}$

$M = 40 - 35e^{-t/4}$

4.



Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Solutions:

a) $f'(x) > 0$ on (2, 6) and $f'(x) > 0$ on (6, 8).
$f'(x)$ does not change sign at $x = 6$, so there is neither a relative maximum nor a relative minimum at this location.

b) The graph of f is concave down on $(-2, 0)$ and (4, 6) because f' is decreasing on these intervals.

c) Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x \to 2} f(x) = f(2) = 1$.

$\lim_{x \to 2} (6f(x) - 3x) = 6 \cdot 1 - 3 \cdot 2 = 0$

$\lim_{x \to 2} (x^2 - 5x + 6) = 0$

Because $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$ is of indeterminate form $\dfrac{0}{0}$, L'Hospital's Rule can be applied.
Using L'Hospital's Rule,

$$\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{6f'(x) - 3}{2x - 5} = \frac{6 \cdot 0 - 3}{2 \cdot 2 - 5} = 3.$$

d) The function f is continuous on $[-2, 8]$, so the candidates for the location of an absolute minimum for f are $x = -2$, $x = -1$, $x = 2$, $x = 6$, and $x = 8$.

x	$f(x)$
-2	3
-1	4
2	1
6	$7 - \pi$
8	$11 - 2\pi$

The absolute minimum value of f is $f(2) = 1$.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

5. The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Solutions:

a) $h'(x) = f'(g(x)) \cdot g'(x)$
$h'(7) = f'(g(7)) \cdot g'(7)$
$= f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$

b) $k''(x) = 2f(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$
$k''(4) = 2f(4) \cdot f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
$= 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2 = -72 + 32 = -40$
The graph of k is concave down at the point where $x = 4$ because $k''(4) < 0$ and k'' is continuous.

c) $m(2) = 5 \cdot 8 + \int_0^2 f'(t)dt = 40 + (f(2) - f(0)) = 40 + (7 - 10) = 37$

d) $m'(x) = 15x^2 + f'(x)$
$m'(2) = 15 \cdot 4 + f'(2) = 60 + (-8) = 52$
The graph of m is increasing at $x = 2$ because $m'(2) > 0$.

6. Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$.

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\dfrac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\dfrac{dx}{dt} = \dfrac{2}{3}$ unit per second. What is the value of $\dfrac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

Solutions:

a) $\dfrac{d}{dx}(6xy) = \dfrac{d}{dx}\left(2 + y^3\right) \Rightarrow 6y + 6x\dfrac{dy}{dx} = 3y^2\dfrac{dy}{dx}$

$\Rightarrow 2y = \dfrac{dy}{dx}\left(y^2 - 2x\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$

b) For the line tangent to the curve to be horizontal, it is necessary that $2y = 0$ (so $y = 0$) and that $y^2 - 2x \neq 0$.

Substituting $y = 0$ into $6xy = 2 + y^3$ yields the equation $6x \cdot 0 = 2$, which has no solution.

Therefore, there is no point on the curve at which the line tangent to the curve is horizontal.

c) For a line tangent to this curve to be vertical, it is necessary that $2y \neq 0$ and that $y^2 - 2x = 0$ (so $x = \dfrac{y^2}{2}$).

Substituting $x = \dfrac{y^2}{2}$ into $6xy = 2 + y^3$ yields the equation $3y^2 \cdot y = 2 + y^3 \Rightarrow 2y^3 = 2 \Rightarrow y = 1$.

Substituting $y = 1$ in $6xy = 2 + y^3$ yields $6x = 2 + 1$, or $x = \dfrac{1}{2}$.

The tangent line to the curve is vertical at the point $\left(\dfrac{1}{2}, 1\right)$.

d) $6y\dfrac{dx}{dt} + 6x\dfrac{dy}{dt} = 0 + 3y^2\dfrac{dy}{dt}$

At the point $(x, y) = \left(\dfrac{1}{2}, -2\right)$,

$$6(-2)\left(\frac{2}{3}\right)+6\left(\frac{1}{2}\right)\frac{dy}{dt}=3(-2)^2\frac{dy}{dt}$$

$$\Rightarrow -8+3\frac{dy}{dt}=12\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt}=-\frac{8}{9}\text{ unit per second}$$

Problems adapted from the College Board released tests.