t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

a. Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.

b. Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

c. For $12 \leq t \leq 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$.

Show the setup for your calculations.

d. For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100-t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate.

Give a reason for your answer.

Solutions:

a.

$$C'(5) \approx \frac{C(7) - C(3)}{7-3} = \frac{69-85}{4} = -4 \text{ degrees Celsius per minute}$$

b.

$$\int_0^{12} C(t)\,dt \approx (3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7)$$
$$= 3\cdot 100 + 4\cdot 85 + 5\cdot 69 = 985$$

$\frac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature of the coffee (in degrees Celsius) over the interval from $t = 0$ to $t = 12$.

c.

$$C(20) = C(12) + \int_{12}^{20} C'(t)\,dt$$

$$= 55 - 14.670812 = 40.329188$$

The temperature of the coffee at time $t = 20$ is 40.329 degrees Celsius.

d. Because $C''(t) > 0$ on the interval $12 < t < 20$, the rate of change in the temperature of the coffee, $C'(t)$, is increasing on this interval.
That is, on the interval $12 < t < 20$, the temperature of the coffee is changing at an increasing rate.

2. A particle moves along the x-axis so that its velocity at time $t \geq 0$ is given by $v(t) = \ln(t^2 - 4t + 5) - 0.2t$.

a. There is one time, $t = t_R$, in the interval $0 < t < 2$ when the particle is at rest (not moving). Find t_R. For $0 < t < t_R$, is the particle moving to the right or to the left? Give a reason for your answer.

b. Find the acceleration of the particle at time $t = 1.5$. Show the setup for your calculations. Is the speed of the particle increasing or decreasing at time $t = 1.5$? Explain your reasoning.

c. The position of the particle at time t is $x(t)$, and its position at time $t = 1$ is $x(1) = -3$. Find the position of the particle at time $t = 4$. Show the setup for your calculations.

d. Find the total distance traveled by the particle over the interval $1 \leq t \leq 4$. Show the setup for your calculations.

Solutions:

a.

$$v(t) = 0 \implies t = 1.425610$$

Therefore, the particle is at rest (not moving) at $t_R = 1.426$ (or 1.425).
For $0 < t < t_R$, $v(t) > 0$. Therefore, the particle is moving to the right on that interval.

b.

$$a(1.5) = v'(1.5) = -1$$

The acceleration of the particle at time $t = 1.5$ is -1 (or -0.999).
$v(1.5) = -0.076856 < 0$
Because $a(1.5)$ and $v(1.5)$ have the same sign, the speed is increasing at time $t = 1.5$.

c.

$$x(4) = x(1) + \int_1^4 v(t)\,dt$$
$$= -3 + 0.197117 = -2.802883$$

The position of the particle at time $t = 4$ is -2.803 (or -2.802).

d.

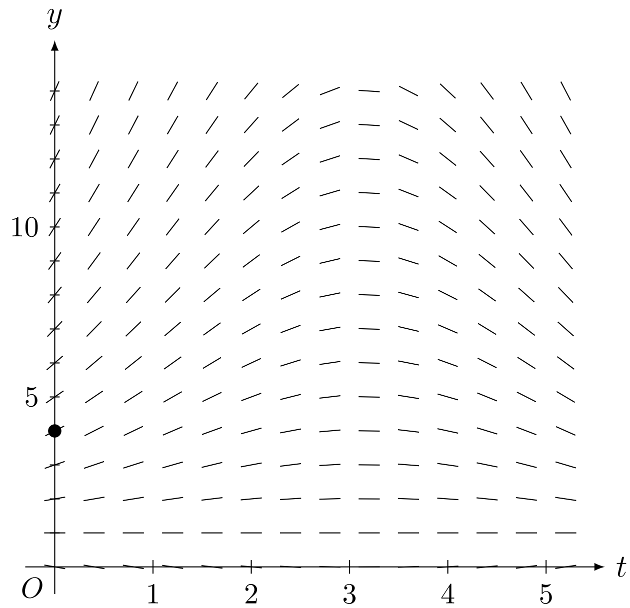
$$\int_1^4 |v(t)|\,dt$$
$$\int_1^{1.425} v(t)\,dt - \int_{1.425}^{2.883} v(t)\,dt + \int_{2.883}^{4} v(t)\,dt$$
$$\int_1^{1.426} v(t)\,dt - \int_{1.426}^{2.883} v(t)\,dt + \int_{2.883}^{4} v(t)\,dt$$

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation

$$\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right),$$

where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

a. A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



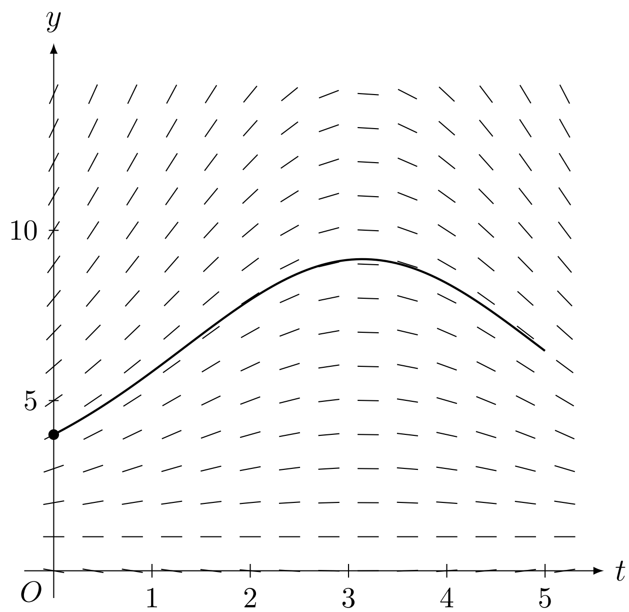
b. For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

c. Use separation of variables to find $y = H(t)$, the particular solution to the differential equation

$$\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right) \text{ with initial condition } H(0) = 4.$$

Solutions:

a.



b. Because $H(t) > 1$, then $\frac{dH}{dt} = 0$ implies $\cos\left(\frac{t}{2}\right) = 0$.
This implies that $t = \pi$ is a critical point.
For $0 < t < \pi$, $\frac{dH}{dt} > 0$ and for $\pi < t < 5$, $\frac{dH}{dt} < 0$. Therefore, $t = \pi$ is the location of a relative maximum value of H.

c.

$$\frac{dH}{H-1} = \frac{1}{2}\cos\left(\frac{t}{2}\right) dt$$

$$\int \frac{dH}{H-1} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right) dt$$

$$\implies \ln|H-1| = \sin\left(\frac{t}{2}\right) + C$$

Using $H(0) = 4$:

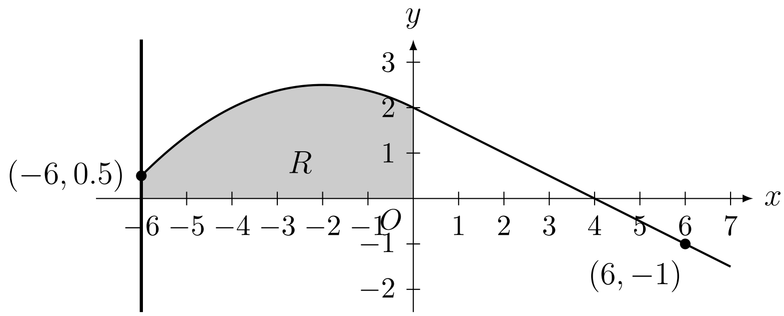
$$\ln|4-1| = \sin(0) + C \implies C = \ln 3$$

Because $H(0) = 4$, $H > 1$, so $|H-1| = H-1$.

$$\ln(H-1) = \sin\left(\frac{t}{2}\right) + \ln 3$$

$$H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$$

$$H(t) = 1 + 3e^{\sin(t/2)}$$



Graph of f

4. The graph of the differentiable function f, shown for $-6 \leq x \leq 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \leq x \leq 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

a. The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

b. For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.

c. The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Solutions:

a.

$$g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12$$

$$g(4) = \int_0^4 f(t)\,dt = \frac{1}{2} \cdot 4 \cdot 2 = 4$$

$$g(6) = \int_0^6 f(t)\,dt = \frac{1}{2} \cdot 4 \cdot 2 - \frac{1}{2} \cdot 2 \cdot 1 = 3$$

b.

$$g'(x) = f(x)$$

$$g'(x) = f(x) = 0 \implies x = 4$$

Therefore, the graph of g has a critical point at $x = 4$.

c.

$$h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$$

$$h'(x) = f'(x), \text{ so } h'(6) = f'(6) = -\frac{1}{2}.$$

$$h''(x) = f''(x), \text{ so } h''(6) = f''(6) = 0.$$

5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3+4y}$.

a. There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.

b. Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

c. The curve intersects the positive x-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

d. For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?

Solutions:

a.

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,4)} = \frac{-2(2)}{3+4(4)} = -\frac{4}{19}$$

$$y \approx 4 - \frac{4}{19}(3-2) = \frac{72}{19}$$

b.

$$\frac{dy}{dx} = \frac{-2x}{3+4y} = 0 \implies x = 0$$

And so, if the horizontal line $y = 1$ is tangent to the curve, the point of tangency must be $(0, 1)$.
However, the point $(0, 1)$ is not on the curve, because $0^2 + 3 \cdot 1 + 2 \cdot 1^2 = 5 \neq 48$.
Therefore, the horizontal line $y = 1$ is not tangent to the curve.

c. At the point $(\sqrt{48}, 0)$, the slope of the line tangent to the curve is $\frac{dy}{dx} = \frac{-2\sqrt{48}}{3+4(0)}$.
The denominator of $\frac{dy}{dx}$ is $3 + 4(0)$, which does not equal 0.
Therefore, the line tangent to the curve at this point is not vertical.

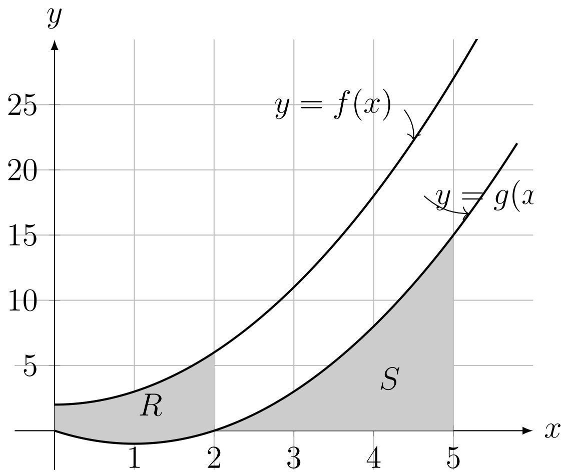
d.

$$3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = -2$$

$$3(2)^2(-2) + 2(4)(-2) + 2(2)\frac{dx}{dt} = 0 \implies \frac{dx}{dt} = \frac{40}{4} = 10$$

The rate of change with respect to time in the x-coordinate is 10 units per second.



6. The functions f and g are defined by $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$, as shown in the graph.

a. Let R be the region bounded by the graphs of f and g, from $x = 0$ to $x = 2$, as shown in the graph. Write, but do not evaluate, an integral expression that gives the area of region R.

b. Let S be the region bounded by the graph of g and the x-axis, from $x = 2$ to $x = 5$, as shown in the graph. Region S is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis is a rectangle with height equal to half its base in region S. Find the volume of the solid. Show the work that leads to your answer.

c. Write, but do not evaluate, an integral expression that gives the volume of the solid generated when region S, as described in part (b), is rotated about the horizontal line $y = 20$.

Solutions:

a.

$$\text{Area} = \int_0^2 (f(x) - g(x))\, dx$$

b.

$$\text{Volume} = \int_2^5 \frac{1}{2}(g(x))^2\, dx = \int_2^5 \frac{1}{2}(x^2 - 2x)^2\, dx$$

$$= \frac{1}{2}\int_2^5 (x^4 - 4x^3 + 4x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_2^5$$

$$= \frac{1}{2}\left[\left(\frac{5^5}{5} - 5^4 + \frac{500}{3}\right) - \left(\frac{32}{5} - 16 + \frac{32}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{500}{3} - \frac{16}{15}\right) = \frac{414}{5}$$

c.

$$\begin{aligned}
\text{Volume} &= \pi \int_2^5 \left[(20^2) - (20 - g(x))^2\right] \, dx \\
&= \pi \int_2^5 \left[400 - (20 - g(x))^2\right] \, dx \\
&= \pi \int_2^5 \left[400 - (20 - (x^2 - 2x))^2\right] \, dx
\end{aligned}$$

Problems adapted from the College Board released tests.