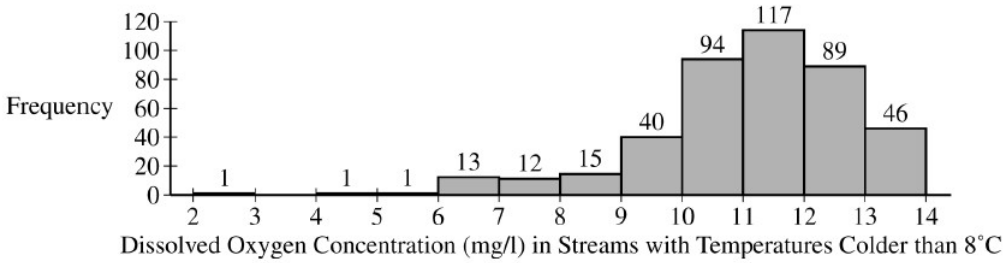
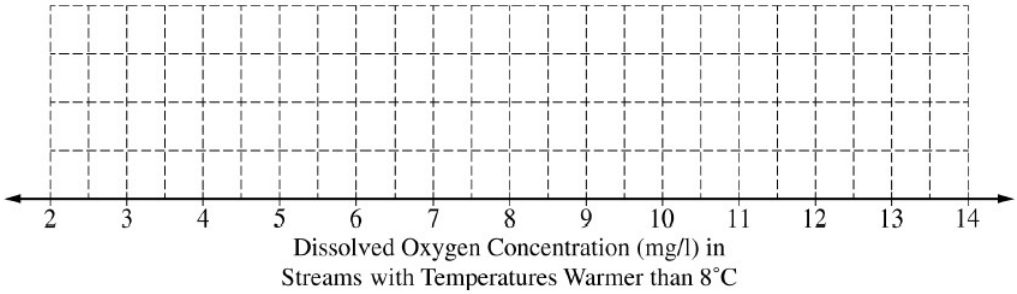
1. As part of a study on the chemistry of Alaskan streams, researchers took water samples from many streams with temperatures colder than 8°C and from many streams with temperatures warmer than 8°C. For each sample, the researchers measured the dissolved oxygen concentration, in milligrams per liter (mg/l).



(A) The researchers constructed the histogram shown for the dissolved oxygen concentration in streams from the sample with water temperatures colder than 8°C. Based on the histogram, describe the distribution of dissolved oxygen concentration in streams with water temperatures colder than 8°C.

Min	Q1	Median	Q3	Max	Mean	Std. Dev.
2.10	4.39	5.43	6.12	13.45	5.54	1.64

(B) The researchers computed the summary statistics shown in the table for the dissolved oxygen concentration in streams from the sample with water temperatures warmer than 8°C. Use the summary statistics to construct a box plot for the dissolved oxygen concentration in streams with water temperatures warmer than 8°C. Do not indicate outliers.



(C) The researchers believe that streams with higher dissolved oxygen concentration are generally healthier for wildlife. Which streams are generally healthier for wildlife, those with water temperature colder than 8°C or those with water temperature warmer than 8°C? Using characteristics of the distribution of dissolved oxygen concentration for temperatures colder than 8°C and characteristics of the distribution of dissolved oxygen concentration for temperatures warmer than 8°C, justify your answer.

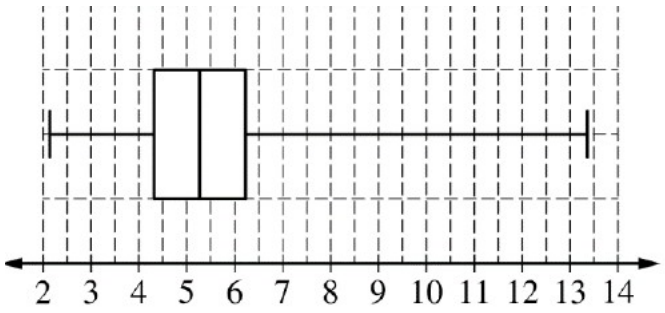
Solution:

(A) The histogram of dissolved oxygen concentration in Alaskan streams with water temperatures colder than 8°C is unimodal and skewed left with a median between 11 and 12 mg/l.

The first quartile is in the bin from 10-11 mg/l and the third quartile is in the bin from 12-13 mg/l, so the IQR is approximately 2 mg/l.

There do not appear to be any high outliers, but there are several potential low outliers because the values in the 2-3, 4-5, and 5-6 bins are all certainly more than 1.5 IQR below the first quartile.

(B)



(C) If the researchers' belief is correct, then streams with water temperature colder than 8°C are healthier for wildlife.

The distribution of dissolved oxygen concentration for colder streams has a higher center because its median (between 11 mg/l and 12 mg/l) is larger than the median for warmer streams (5.43 mg/l).

The shape of the distribution of dissolved oxygen concentration for colder streams is different from the shape of the distribution for warmer streams. The distribution of values of dissolved oxygen concentration for colder streams is skewed to the left but the distribution of values for warmer streams is skewed to the right.

Both distributions have a similar spread because they both have similar IQR values — approximately 2 mg/l for the colder streams and 1.73 mg/l for the warmer streams.

2. A developer wants to know whether adding fibers to concrete used in paving driveways will reduce the severity of cracking, because any driveway with severe cracks will have to be repaired by the developer. The developer conducts a completely randomized experiment with 60 new homes that need driveways. Thirty of the driveways will be randomly assigned to receive concrete that contains fibers, and the other 30 driveways will receive concrete that does not contain fibers. After one year, the developer will record the severity of cracks in each driveway on a scale of 0 to 10, with 0 representing not cracked at all and 10 representing severely cracked.

(A) Based on the information provided about the developer's experiment, identify each of the following.

- Experimental units
- Treatments
- Response variable

(B) Describe an appropriate method the developer could use to randomly assign concrete that contains fibers and concrete that does not contain fibers to the 60 driveways.

Suppose the developer finds that there is a statistically significant reduction in the mean severity of cracks in driveways using the concrete that contains fibers compared to the driveways using concrete that does not contain fibers.

(C) In terms of the developer's conclusion, what is the benefit of randomly assigning the driveways to either the concrete that contains fibers or the concrete that does not contain fibers?

Solution:

(A) Experimental units: 60 driveways.

Treatments: Concrete with fibers and concrete without fibers.

Response variable: Rating of the severity of the cracks after one year, on a scale of 0 to 10.

(B) Number the 60 driveways from 01 to 60. Using a random number generator, generate two-digit integers between 01 and 60. Ignore 00 and any number greater than 60 until 30 unique numbers are obtained. Assign the driveways with those 30 unique numbers to receive concrete with fibers and the remaining 30 driveways to receive concrete without fibers.

(C) The results were statistically significant, and because the driveways were randomly assigned to either the concrete with the fibers or the concrete without the fibers, there is evidence the treatment (type of concrete) caused the response (rating of severity of cracks).

3. Bath fizzies are mineral tablets that dissolve and create bubbles when added to bathwater. In order to increase sales, the Fizzy Bath Company has produced a new line of bath fizzies that have a cash prize in every bath fizzy. Let the random variable, X, represent the dollar value of the cash prize in a bath fizzy. The probability distribution of X is shown in the table.

Cash prize, x	\$1	\$5	\$10	\$20	\$50	\$100
Probability of cash prize, $P(X = x)$	0.2	0.05	0.05	0.05	0.01	0.01

(A) Based on the probability distribution of X, answer the following. Show your work.

(i) Calculate the proportion of bath fizzies that contain \$1.

(ii) Calculate the proportion of bath fizzies that contain at least \$10.

(B) Based on the probability distribution of X, calculate the probability that a randomly selected bath fizzy contains \$100, given that it contains at least \$10. Show your work.

(C) Based on the probability distribution of X, calculate and interpret the expected value of the distribution of the cash prize in the bath fizzies. Show your work.

(D) The Fizzy Bath Company would like to sell the bath fizzies in France, where the currency is euros. Suppose the conversion rate for dollars to euros is 1 dollar = 0.89 euros. Using your expected value from part (c), calculate the expected value, in euros, of the distribution of the cash prize in the bath fizzies. Show your work.

Solution:
(A) The random variable X is the dollar value of the cash prize in a bath fizzy.
(i) The proportion of bath fizzies containing \$1 is equal to the $P(X = \$1)$ and

$$P(X = \$1) = 1 - (0.2 + 0.05 + 0.05 + 0.01 + 0.01) = 0.68.$$

(ii) The proportion of bath fizzies that contain at least \$10 is equal to the $P(X \geq \$10)$ and

$$P(X \geq \$10) = 0.05 + 0.05 + 0.01 + 0.01 = 0.12.$$

(B) Given a bath fizzy contains at least \$10, then the probability that it contains \$100 is

$$P(X = \$100 \mid X \geq \$10) = \frac{0.01}{0.12} \approx 0.0833.$$

(C) The expected value of the distribution of X is

$$\begin{aligned} E(X) &= 1(0.68) + 5(0.2) + 10(0.05) \\ &\quad + 20(0.05) + 50(0.01) + 100(0.01) = \$4.68. \end{aligned}$$

The expected value is the mean of the cash prizes that result from the long run of many, many trials of randomly selecting bath fizzies and determining the amount each contains.

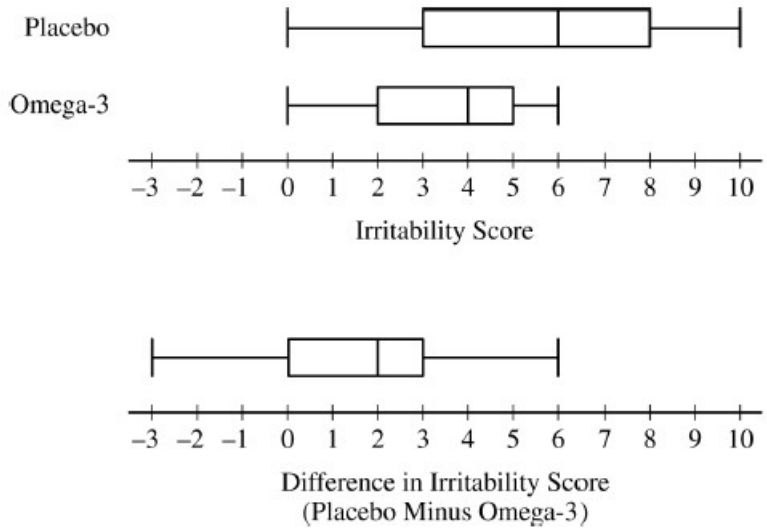
(D) The expected value of the distribution of X in euros is $4.68(0.89) \simeq 4.17$ euros.

4. A medical researcher completed a study comparing an omega-3 fatty acids supplement to a placebo in the treatment of irritability in patients with a certain medical condition. Nineteen patients with the medical condition volunteered to participate in the study. The study was conducted using the following weekly schedule.

- Week 1: Each patient took a randomly assigned treatment, omega-3 supplement or placebo.
- Week 2: The patients did not take either the omega-3 supplement or the placebo. This was necessary to reduce the possibility of any carryover effect from the assigned treatment taken during week 1.
- Week 3: Each patient took the treatment, omega-3 supplement or placebo, that they did not take during week 1.

At the end of week 1 and week 3, each patient's irritability was given a score on a scale of 0 to 10, with 0 representing no irritability and 10 representing the highest level of irritability.
For each patient, the two irritability scores and the difference in their scores (placebo minus omega-3) were recorded. The results are summarized in the table and boxplots.

	n	Mean	Standard Deviation
Placebo	19	5.421	2.987
Omega-3	19	3.632	1.739
Difference (placebo minus omega-3)	19	1.789	2.485



The researcher claims the omega-3 supplement will decrease the mean irritability score of all patients with the medical condition similar to the volunteers who participated in the study. Is there convincing statistical evidence to support the researcher's claim at a significance level of $\alpha = 0.05$? Complete the appropriate inference procedure to support your answer.

Solution:

Section 1 Let μ_d represent the true mean difference (placebo minus omega-3) of irritability scores for all people with this medical condition.

The null hypothesis is H_0: $\mu_d = 0$ and the alternative hypothesis is H_a: $\mu_d > 0$.

The appropriate inference procedure is a matched pairs t-test for a mean difference.

Section 2 The independence condition for performing a paired t-test for a mean difference is satisfied because the data were obtained from a randomized experiment where the week in which the patient received the treatment was randomly assigned.

The sampling distribution of the mean difference must be approximately normal. Although the sample size is less than 30 ($n = 19$), this is satisfied because the boxplot for the sample differences shows an approximately symmetric distribution with no outliers.

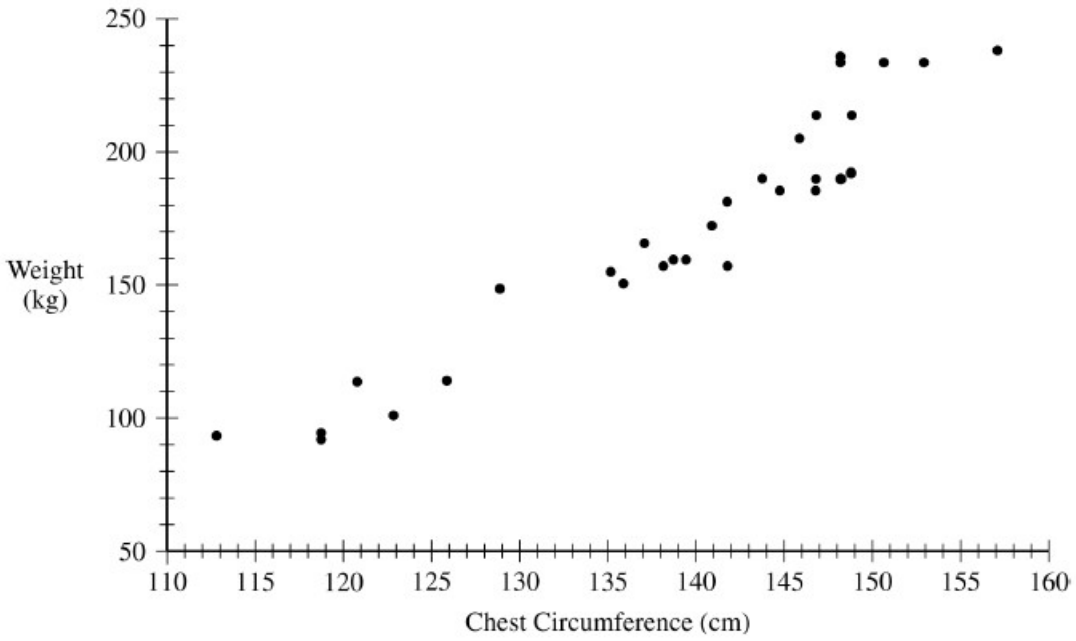
The value of the test statistic is:

$$t = \frac{\overline{x}_d - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{1.789 - 0}{\frac{2.485}{\sqrt{19}}} \approx 3.138$$

Using 18 degrees of freedom, the corresponding p-value is $P(t > 3.138) \approx 0.0028$.

Section 3 Because the p-value ≈ 0.0028 is less than the significance level, $\alpha = 0.05$, the null hypothesis should be rejected. The data provide convincing statistical evidence that for patients similar to those in the study, the true mean difference (placebo minus omega-3) in irritability scores for people with this medical condition is greater than zero. This suggests the omega-3 fatty acids are helpful in reducing irritability scores in people with this medical condition.

5. Wildlife biologists are interested in the health of tule elk, a species of deer found in California. An important measurement of tule elk health is their weight. The weight of a tule elk is difficult to measure in the wild. However, chest circumference, which is believed to be related to the weight of a tule elk, can easily be measured from a safe distance using a harmless laser. A study was done to investigate whether chest circumference, in centimeters (cm), could be used to accurately estimate the weight, in kilograms (kg), of male tule elk. For the study, wildlife biologists captured 30 male tule elk, measured their chest circumference and weight, and then released the elk. The data for the 30 male tule elk are shown in the scatterplot.



(A) Describe the relationship between chest circumference and weight of male tule elk in context.

Following is the equation of the least-squares regression line relating chest circumference and weight for male tule elk.

$$\text{predicted weight} = -350.3 + 3.7455(\text{chest circumference})$$

(B) The weight of one male tule elk with a chest circumference of 145.9 cm is 204.3 kg.

(i) Using the equation of the least-squares regression line, calculate the predicted weight for this male tule elk. Show your work.

(ii) Calculate the residual for this male tule elk. Show your work.

The equation of the least-squares regression line relating chest circumference and weight for male tule elk is repeated here.

$$\text{predicted weight} = -350.3 + 3.7455(\text{chest circumference})$$

(C) Interpret the slope of the least-squares regression line in context.

(D) The sambar, another species of deer, is similar in size to the tule elk. The slope of the population regression line relating chest circumference and weight for all male sambars is 4.5 kilograms per centimeter. A wildlife biologist wants to determine whether the slope of the population regression line for male tule elk is different than that for male sambars. Let β represent the slope of the population regression line for male tule elk. The wildlife biologist conducted a test of the following hypotheses using the sample of 30 tule elk.

$$H_0 : \beta = 4.5$$
$$H_a : \beta \neq 4.5$$

The test statistic was calculated to be 3.408. Assume all conditions for inference were met.

(i) Determine the p-value of the test.

(ii) At a significance level of $\alpha = 0.05$, what conclusion should the wildlife biologist make regarding the slope of the population regression line for male tule elk? Justify your response.

Solution:

(A) The scatterplot reveals a strong, positive, roughly linear association between the chest circumference and weight of tule elk. There are no points that seriously deviate from the straight-line pattern of the points in the plot.

(B)
(i) The predicted weight of a male tule elk with a chest circumference of 145.9 cm is $-350.3 + 3.7455(145.9) \approx 196.17$ kg.

(ii) The residual for a male tule elk with a chest circumference of 145.9 cm with an actual weight of 204.3 kg is $204.3 - 196.17 \approx 8.13$ kg.

(C) The value of the slope of the least-squares regression line is 3.7455. This value indicates that the predicted weight of a tule elk increases by 3.7455 kilograms for each additional centimeter of chest circumference.

(D)
(i) The degrees of freedom for the test of slope are $n - 2 = 30 - 2 = 28$. The t-table shows that for 28 degrees of freedom, the p-value for a one-sided test would be 0.001. Because this is a two-sided test, the p-value is $(2)(0.001) = 0.002$.

(ii) Because the p-value $= 0.002$ is less than $\alpha = 0.05$, reject the null hypothesis. There is sufficient statistical evidence that the population slope for the linear regression of weight vs. chest circumference for male tule elk is different from 4.5 kg/cm.

6. A jewelry company uses a machine to apply a coating of gold on a certain style of necklace. The amount of gold applied to a necklace is approximately normally distributed. When the machine is working properly, the amount of gold applied to a necklace has a mean of 300 milligrams (mg) and standard deviation of 5 mg.

(A) A necklace is randomly selected from the necklaces produced by the machine. Assuming that the machine is working properly, calculate the probability that the amount of gold applied to the necklace is between 296 mg and 304 mg.

The jewelry company wants to make sure the machine is working properly. Each day, Cleo, a statistician at the jewelry company, will take a random sample of the necklaces produced that day. Each selected necklace will be melted down and the amount of the gold applied to that necklace will be determined. Because a necklace must be destroyed to determine the amount of gold that was applied, Cleo will use random samples of size $n = 2$ necklaces.

Cleo starts by considering the mean amount of gold being applied to the necklaces. After Cleo takes a random sample of $n = 2$ necklaces, she computes the sample mean amount of gold applied to the two necklaces.

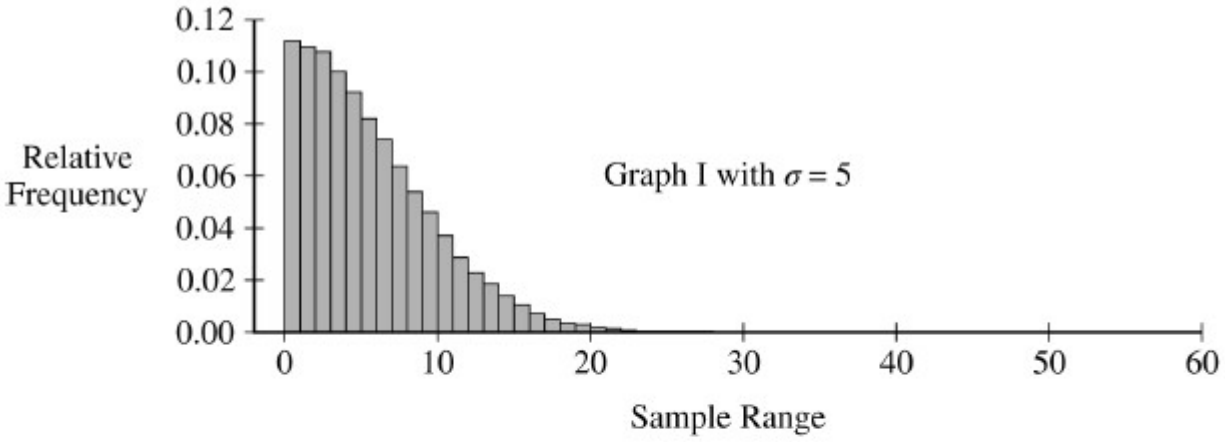
(B) Suppose the machine is working properly with a population mean amount of gold being applied of 300 mg and a population standard deviation of 5 mg.

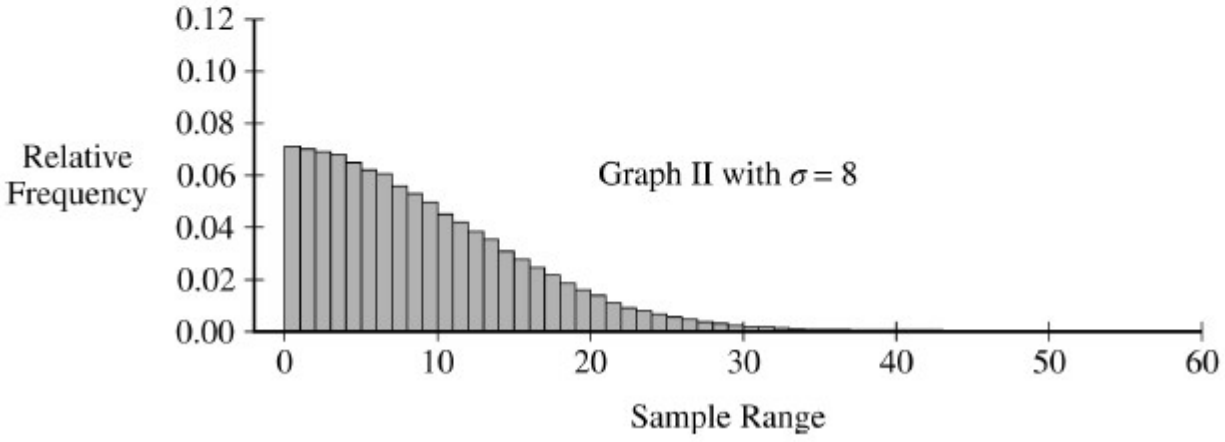
(i) Calculate the probability that the sample mean amount of gold applied to a random sample of $n = 2$ necklaces will be greater than 303 mg.

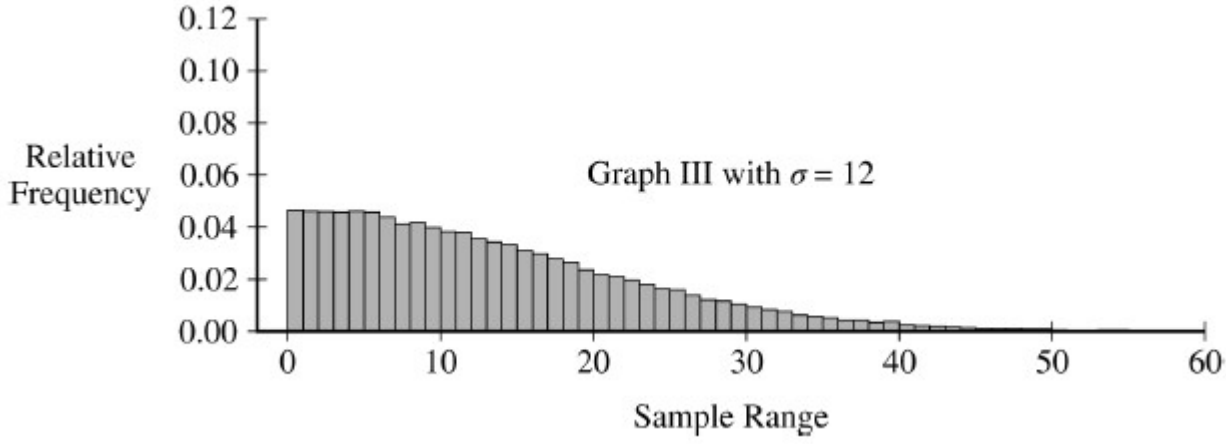
(ii) Suppose Cleo took a random sample of $n = 2$ necklaces that resulted in a sample mean amount of gold applied of 303 mg. Would that result indicate that the population mean amount of gold being applied by the machine is different from 300 mg? Justify your answer without performing an inference procedure.

Now, Cleo will consider the variation in the amount of gold the machine applies to the necklaces. Because of the small sample size, $n = 2$, Cleo will use the sample range of the data for the two randomly selected necklaces, rather than the sample standard deviation.

Cleo will investigate the behavior of the range for samples of size $n = 2$. She will simulate the sampling distribution of the range of the amount of gold applied to two randomly sampled necklaces. Cleo generates $100,000$ random samples of size $n = 2$ independent values from a normal distribution with mean $\mu = 300$ and standard deviation $\sigma = 5$. The range is calculated for the two observations in each sample. The simulated sampling distribution of the range is shown in Graph I. This process is repeated using $\sigma = 8$, as shown in Graph II, and again using $\sigma = 12$, as shown in Graph III.







(C) Use the information in the graphs to complete the following.

(i) Describe the sampling distribution of the sample range for random samples of size $n = 2$ from a normal distribution with standard deviation $\sigma = 5$, as shown in Graph I.

(ii) Describe how the sampling distribution of the sample range for samples of size $n = 2$ changes as the value of the population standard deviation σ increases.

Recall that Cleo needs to consider both the mean and standard deviation of the amount of gold applied to necklaces to determine whether the machine is working properly. Suppose that one month later, Cleo is again checking the machine to make sure it is working properly. Cleo takes a random sample of 2 necklaces and calculates the sample mean amount of gold applied as 303 mg and the sample range as 10 mg.

(D) Recall that the machine is working properly if the amount of gold applied to the necklaces has a mean of 300 mg and standard deviation of 5 mg.

(i) Consider Cleo's range of 10 mg from the sample of size $n = 2$. If the machine is working properly with a standard deviation of 5 mg, is a sample range of 10 mg unusual? Justify your answer.

(ii) Do Cleo's sample mean of 303 mg and range of 10 mg indicate that the machine is not working properly? Explain your answer.

Solutions:

(A) Let X represent the amount of gold applied to a necklace randomly selected from necklaces produced with this machine. The random variable X has an approximately normal distribution with mean 300 mg and standard deviation 5 mg.

Then,

$$\begin{aligned} P(296 < X < 304) &= P(X < 304) - P(X \leq 296) \\ &= P\left(Z < \frac{304 - 300}{5}\right) - P\left(Z \leq \frac{296 - 300}{5}\right) \\ &= P(Z < 0.8) - P(Z \leq -0.8) \\ &\approx 0.7881 - 0.2119 \approx 0.5763. \end{aligned}$$

(B) (i) If the machine is working properly, and the sample mean amount of gold, $\overline{X}$, has a sampling distribution that follows a normal distribution with mean 300 mg and standard deviation

$$\frac{5}{\sqrt{2}} \approx 3.5355 \text{ mg, then}$$

$$\begin{aligned} P(\overline{X} > 303) &= P\left(Z > \frac{303 - 300}{\frac{5}{\sqrt{2}}}\right) \\ &\approx P(Z > 0.8485) \\ &\approx 0.198. \end{aligned}$$

(ii) Observing a sample mean amount of 303 mg would not provide convincing evidence that the population mean amount of gold being applied by the machine is something other than 300 mg because

the probability of observing a sample mean that differs from 300 mg by 3 mg or more is large, around $0.198(2) = 0.396$.

(C) (i) The sampling distribution of the sample range for random samples of size $n = 2$ from a normal distribution with standard deviation $\sigma = 5$ is skewed to the right. Almost all values of the simulated ranges are between 0 mg and about 25 mg and the center of the distribution is about 6 mg.

(ii) As the value of the population standard deviation increases, the variation (spread) in the distribution of the sample range increases and the mean of the distribution of the sample range also increases.

(D) (i) No, a sample range of 10 mg is not unusual if the machine is working properly with a standard deviation of 5 mg.

Although observing a sample range around 10 mg or greater is much more likely if the population standard deviation is 8 mg or 12 mg than when the population standard deviation is 5 mg, the graph of the sampling distribution of the sample range for samples of size 2 from a normal distribution with $\sigma = 5$ mg indicates that 10 mg is not an unusual value for the range when $\sigma = 5$ mg. There is about a 20% chance that a random sample of two necklaces would yield a range of 10 mg or more when the machine is working properly.

(ii) No, Cleo's sample mean of 303 mg and range of 10 mg do not indicate that the machine is not working properly. As noted in part (b-i), the probability that the sample mean would be equal to or greater than 303 mg when the machine is working properly is almost 20% so having a sample mean of 303 mg is not unusual. Furthermore, it is less than one standard deviation, $\frac{5}{\sqrt{2}} = 3.5355$ mg, away from 300 mg. As indicated in part (d-i), the probability of a range of 10 mg or greater when the population standard deviation is 5 mg is also about 20%, so not unusual. There is not statistically significant evidence to show the machine is not working properly.

Problems adapted from the College Board released tests.